

A symmetric difference-differential Lax pair for Painlevé VI

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ABSTRACT. We present a Lax pair for the sixth Painlevé equation arising as a continuous isomonodromic deformation of a system of linear difference equations with an additional symmetry structure. We call this a symmetric difference-differential Lax pair. We show how the discrete isomonodromic deformations of the associated linear problem gives us a discrete version of the fifth Painlevé equation. By considering degenerations we obtain symmetric difference-differential Lax pairs for the fifth Painlevé equation and the various degenerate versions of the third Painlevé equation.

1. Introduction

The Painlevé equations are second order nonlinear differential equations whose only movable singularities are poles [5]. The most general case in the classification of the Painlevé equations is the sixth Painlevé equation, written in standard form as

$$(P_{VI}) \quad \frac{d^2 y}{dt^2} = \frac{1}{2} \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t} \right) \left(\frac{dy}{dt} \right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t} \right) \frac{dy}{dt} \\ + \frac{y(y-1)(y-t)}{(t-1)^2 t^2} \left(\alpha + \frac{\beta t}{y^2} + \frac{\gamma(t-1)}{(y-1)^2} + \frac{\delta(t-1)t}{(y-t)^2} \right),$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ are parameters. For special values of the parameters the solutions of (P_{VI}) have been used to express classes of Einstein metrics [29], correlation functions in the 2D Ising model [15], express the eigenvalue distributions for certain ensembles of random matrices [8] and characterize the reductions of nonlinear wave equations [20] and the self-dual Yang-Mills equations [19]. As a mathematical object, (P_{VI}) possesses a group of Bäcklund transformations that is of affine Weyl type $D_4^{(1)}$ [25], possesses solutions expressible in terms of Gauss's hypergeometric function [13] and admits a surface of initial conditions with a rich geometric structure [12].

Identifying the sixth Painlevé equation in applications usually hinges upon making a correspondence with a known Lax pair. This is usually done by showing that a system arises as an isomonodromic deformation of an associated linear problem of the right type. The most celebrated example arises as the isomonodromic deformations of a second order linear differential equation with four Fuchsian singularities [9, 10]. This makes understanding the Lax pairs of (P_{VI}) critically important in

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applications. For this reason, the Lax pairs of (P_{VI}) have been the subject of many works [6, 18, 23].

Finding a Lax pair for a given system can be a difficult or even impossible task. The starting point is usually an ansatz made about the properties of some associated linear problem, one then tries to show that a given system coincides with the system of isomonodromic deformations [16]. This ansatz can be guided by the geometry of the moduli space of linear problems, which may be moduli spaces of systems of linear differential equations (or connections) [3, 11], or moduli spaces of difference equations, which were the subject of an article by one of the authors [27].

The goal of this paper is to present a new Lax pair for (P_{VI}) which has been derived as a *continuous* isomonodromic deformation of a system of linear *difference* equations. We may write this system in matrix form as

$$(1.1a) \quad Y(x+1) = A(x)Y(x),$$

where x is called a spectral variable. The isomonodromic deformations are with respect to a new independent variable t , in which the evolution in t is governed by an auxiliary linear problem of the form

$$(1.1b) \quad \frac{dY(x)}{dt} = B(x)Y(x).$$

We call Lax pairs that take the form (1.1a) and (1.1b) difference-differential Lax pairs. A novel feature of the Lax pair for (P_{VI}) we present is the presence of an additional symmetry; that the solutions of (1.1a) satisfy

$$(1.2) \quad Y(x) = Y(-x - \sigma),$$

where $\sigma \in \mathbb{C}$. We call Lax pairs of this form symmetric difference-differential Lax pairs. By considering degenerations of the Lax pair we present we also obtain symmetric difference-differential Lax pairs for

$$(P_V) \quad \frac{d^2 y}{dt^2} = \left(\frac{1}{2y} + \frac{1}{y-1} \right) \left(\frac{dy}{dt} \right)^2 - \frac{1}{t} \frac{dy}{dt} + \frac{(y-1)^2}{t^2} \left(\alpha y + \frac{\beta}{y} \right) + \frac{\gamma y}{t} - \frac{1}{2} \frac{y(y+1)}{y-1},$$

$$(P_{III}(D_6)) \quad \frac{d^2 y}{dt^2} = \frac{1}{y} \left(\frac{dy}{dt} \right)^2 - \frac{1}{t} \frac{dy}{dt} + \frac{\alpha y^2 + \beta}{t} + \gamma y^3 + \frac{\delta}{y},$$

$$(P_{III}(D_7)) \quad \frac{d^2 y}{dt^2} = \frac{1}{y} \left(\frac{dy}{dt} \right)^2 - \frac{1}{t} \frac{dy}{dt} - \frac{2y^2}{t^2} + \frac{\beta}{4t} - \frac{1}{y},$$

$$(P_{III}(D_8)) \quad \frac{d^2 y}{dt^2} = \frac{1}{y} \left(\frac{dy}{dt} \right)^2 - \frac{1}{t} \frac{dy}{dt} + \frac{y^2}{t^2} - \frac{1}{t}.$$

From the classification of Painlevé equations by their surface of initial conditions [28] the three versions of the third Painlevé equation have surfaces of initial conditions with distinct symmetry groups, hence, should be considered as distinct cases. In particular, $(P_{III}(D_7))$ and $(P_{III}(D_8))$ do not admit the first Painlevé equation as a limit. A degeneration diagram is shown in Figure 1.

Amongst the known Lax pairs for the Painlevé equations appearing in figure 1, we note that there are differential-differential Lax pairs for each of the Painlevé

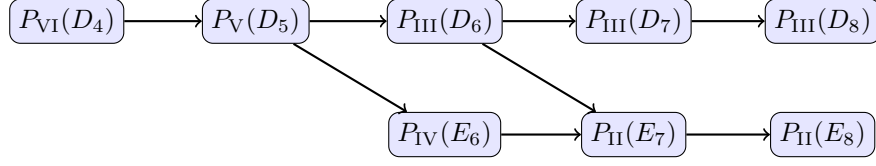


FIGURE 1. The degeneration diagram for the continuous Painlevé equations, including the degenerate cases of the third Painlevé equation. The brackets indicate the affine Weyl symmetry of the rational surface of initial conditions.

equations [14, 24]. However, there are relatively few works that consider continuous isomonodromic deformations of linear difference equations. Difference-differential Lax pairs for P_I , P_{II} , P_{IV} , P_V and P_{VI} all appear in [1] and the difference-differential Lax pairs for the versions of P_{III} only recently appeared [17].

We organize this article as follows: In §2 we present some background on known Lax pairs of the sixth Painlevé equation. In §3 we present our new Lax pair and a system of isomonodromic deformations. In §4 we present a general form of the Lax pair, which makes it clear how one degenerates the Lax pair to give (P_V) , $(P_{III}(D_6))$, $(P_{III}(D_7))$ and $(P_{III}(D_8))$. In §5, we present a simple form of (P_{VI}) in which the limits to versions of (P_V) , $(P_{III}(D_6))$, $(P_{III}(D_7))$ and $(P_{III}(D_8))$ easily follow.

2. Background and motivation

In 1905, it was reported by R. Fuchs that if a scalar linear differential equation with four regular singular points, located at 0, 1, t and ∞ and one apparent singularity at a value y , were to possess a monodromy representation that was independent of the singular point t , then y necessarily satisfies (P_{VI}) . We may transform this to the second order Fuchsian differential equation, giving the following result.

THEOREM 2.1 ([9, 10]). *The isomonodromic deformations of the second order linear differential equation*

$$(2.1) \quad \phi_{xx}(x, t) + \tau_1(x, t)\phi_x(x, t) + \tau_2(x, t)\phi(x, t) = 0,$$

where

$$\begin{aligned} \tau_1(x, t) &= \frac{1 - \kappa_0}{x} + \frac{1 - \kappa_1}{x - 1} + \frac{1 - \kappa_t}{x - t} + \frac{1}{x - y}, \\ \tau_2(x, t) &= \frac{1}{x(x - 1)} \left(\frac{y(y - 1)z}{x - y} - \frac{t(t - 1)H_{VI}}{x - t} + \rho(\kappa_\infty + \rho) \right), \\ H_{VI} &= \frac{y(y - 1)(y - t)}{t(t - 1)} \left(z^2 - \left(\frac{\kappa_0}{y} + \frac{\kappa_1}{y - 1} + \frac{\kappa_t - 1}{y - t} \right) z + \frac{\rho(\kappa_\infty + \rho)}{y(y - 1)} \right), \\ \kappa_0 + \kappa_1 + \kappa_t + \kappa_\infty + 2\rho &= 1. \end{aligned}$$

is equivalent to $y(t)$ satisfying (P_{VI}) where

$$\alpha = \frac{\kappa_\infty^2}{2}, \quad \beta = -\frac{\kappa_0^2}{2}, \quad \gamma = \frac{\kappa_1^2}{2}, \quad \delta = \frac{1 - \kappa_t^2}{2}.$$

PROOF. If we move t in a continuous manner, we may use (2.1) to express the evolution in t in the general form

$$(2.2) \quad \phi_t(x, t) + \sigma_1(x, t)\phi_x(x, t) + \sigma_2(x, t)\phi(x, t) = 0.$$

We can compute $\sigma_1(x, t)$ and $\sigma_2(x, t)$ by comparing the two ways of calculating $\phi_{xxt}(x, t)$; either by taking the derivative in t of (2.1) or the second derivative in x of (2.2). In each case, these may be expressed as linear combinations of $\phi(x, t)$ and $\phi_x(x, t)$. Comparing these expressions is equivalent to Hamilton's equations, given by

$$(2.3) \quad \frac{dy}{dt} = \frac{\partial H_{VI}}{\partial z}, \quad \frac{dz}{dt} = -\frac{\partial H_{VI}}{\partial y}.$$

which is equivalent to $y(t)$ satisfying (P_{VI}) for the above parameters. \square

The equations (2.1) and (2.2) in this calculation constitute a differential-differential Lax pair for (P_{VI}) . Alternatively, one may express the isomonodromic deformation problem in terms of matrices. It was the work of Jimbo, Miwa and Ueno that extended the work on isomonodromic deformations from systems with only Fuchsian singularities to systems of differential equations with higher order singularities. It is in this work we find the following parameterization of a Lax pair for (P_{VI}) in terms of matrices [14].

THEOREM 2.2 ([14]). *Consider the linear system of linear difference equations*

$$(2.4) \quad \frac{dY(x)}{dx} = \left(\frac{A_0}{x} + \frac{A_1}{x-1} + \frac{A_t}{x-t} \right) Y(x),$$

$$A_0 + A_1 + A_t + \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix} = 0,$$

where

$$(2.5) \quad A_i = \begin{pmatrix} z_i + \theta_i & u_i z_i \\ \frac{z_i + \theta_i}{u_i} & -z_i \end{pmatrix},$$

for $i = 0, 1, t$. If $A(x) = (a_{i,j}(x))$ with

$$(2.6) \quad a_{1,2}(x) = \frac{w(x-y)}{x(x-1)(x-t)}, \quad a_{1,1}(y) = z,$$

then moving the parameter t isomonodromically requires that $y = y(t)$ satisfies (P_{VI}) for

$$\alpha = \frac{(\kappa_1 - \kappa_2 - 1)^2}{2}, \quad \beta = -\frac{\theta_0^2}{2}, \quad \gamma = \frac{\theta_1^2}{2}, \quad \delta = -\frac{1 - \theta_t^2}{2}.$$

PROOF. One method of deriving the system of isomonodromic deformations is to apply the Schlesinger equations, which, in this case specialize to

$$(2.7) \quad \frac{dA_0}{dt} = \frac{[A_t, A_0]}{t},$$

$$(2.8) \quad \frac{dA_1}{dt} = \frac{[A_t, A_1]}{t-1},$$

$$(2.9) \quad \frac{dA_t}{dt} = \frac{[A_0, A_t]}{t} + \frac{[A_1, A_t]}{t-1}.$$

This should be analogous to taking the deformation equation to be a linear system of the form (1.1b). The compatibility requires that the mixed derivatives agree, giving the condition

$$(2.10) \quad \frac{\partial A(x)}{\partial t} + A(x)\mathcal{B}(x) = \frac{\partial \mathcal{B}(x)}{\partial x} + \mathcal{B}(x)A(x).$$

In fact, by taking

$$(2.11) \quad \mathcal{B}(x) = -\frac{A_t}{x-t},$$

then the residues of (2.10) at $x = 0$, $x = 1$ and $x = t$ give (2.7), (2.8) and (2.9) respectively. Computing the entries of (2.10) gives

$$\begin{aligned} \frac{dy}{dt} &= \frac{y(y-1)(y-t)}{t(t-1)} \left(2z - \frac{\theta_0}{y} - \frac{\theta_1}{y-1} - \frac{\theta_t-1}{y-t} \right), \\ \frac{dz}{dt} &= \frac{1}{t(t-1)} \left((2(1+t)y - t - 3y^2)z^2 \right. \\ &\quad \left. + ((2y-1-t)\theta_0 + (2y-t)\theta_1 + (2y-1)(\theta_t-1))z - \kappa_1(\kappa_2+1) \right) \end{aligned}$$

Eliminating z from this equation shows y satisfies (P_{VI}) for the given parameters. \square

The equations (2.4) and (1.1b) where $\mathcal{B}(x)$ is specified by (2.11) constitute a differential-differential Lax pair for (P_{VI}) . We consider the scalar Lax pair, (2.1) and (2.2), and the matrix Lax pair, (2.4) and (2.11) to be characteristically the same since it is trivial to convert (2.1) into a matrix form and (2.4) into a scalar form.

To obtain a characteristically different Lax pair, let us consider the above calculations in a geometric framework. A useful geometric relaxation of the notion of a linear system of differential equations is the notion of a connection on a vector bundle, say V . We recover the notion of a matrix differential equation when we restrict our attention the case in which $V = \mathcal{O}_{\mathbb{P}^1}^n$ (copies of sheaves of holomorphic functions on \mathbb{P}^1). In this setting, (P_{VI}) may be interpreted as a flow on the 2-dimensional moduli space of second order Fuchsian linear differential equation with four singular points and specified exponents at those points. This moduli space has the structure of a rational surface with an anticanonical divisor $Y = -\mathcal{K}$ with a decomposition into five irreducible components, one of these components has multiplicity 2. This means the moduli space may be identified as the surface of initial conditions for (P_{VI}) .

Difference equations can also be viewed as maps between the fibres of a vector bundle [2], however, this identification still hinges upon identifying a map as matrix (through trivializing the vector bundle). Both the moduli space of matrices and the moduli space of maps between vector bundles may be interpreted as moduli spaces of sheaves on a smooth projective variety. These relaxations of the difference equations and their moduli spaces have been the subject of a recent study [27]. These moduli spaces can have much the same structure as their continuous counterparts, hence, our motivation was to find a moduli space of difference equations that could also be identified with the surface of initial conditions for (P_{VI}) . The canonical flow on this surface should and does correspond to (P_{VI}) , as we will demonstrate.

One way to find a moduli space systems of linear difference equations of the same type as a given linear system of differential equations is via Z -transform.

This is an invertible transformation that turns a linear differential equation into a linear difference equation. We suppose that a solution of (2.1) is given by a formal biinfinite power series,

$$\phi(x, t) = \sum_{\chi \in \mathbb{Z}} a(\chi, t) x^\chi,$$

then the coefficients, $a(\chi, t)$ satisfy a non-autonomous linear difference in the variable χ . While applying this procedure to (2.1) directly results in a third order equation, we may transform this into a second order differential equation in which the entries are at most quadratic in the new spectral variable. This allows us to obtain a difference equation of the form (1.1a) where

$$A(x) = A_0 + A_1 x + A_2 x^2,$$

with the properties that A_2 has eigenvalues 1 and t , the trace of A_1 is constant and the determinant of $A(x)$ is of quartic. We let a_1, \dots, a_4 be the roots of $\det A(x)$, in which case we write

$$\begin{aligned} (2.12) \quad \det A(x) &= t(x - a_1)(x - a_2)(x - a_3)(x - a_4), \\ &= t(x^4 - \zeta_1 x^3 + \zeta_2 x^2 - \zeta_3 x + \zeta_4), \end{aligned}$$

where ζ_1, \dots, ζ_4 are the elementary symmetric functions. The moduli space of such matrices up to gauge equivalence is a rational surface.

Due to the invertible nature of the Z -transform, and the results of [12], we expect this moduli space to coincide with the space of initial conditions for (P_{VI}) on an open subset of the surface. Exploiting some freedom in how we choose $A(x)$ allows us to simplify the associated linear problem to the following

$$(2.13) \quad A(x) = \begin{pmatrix} (x - z)(x - \tau_1) + y_1 & w(x - z) \\ \frac{\tau_3 x + \tau_4}{w} & t(x - z)(x - \tau_2) + y_2 \end{pmatrix},$$

where we may solve the relations a the coefficients of x in (2.12) by letting

$$(2.14a) \quad \tau_1 = -\sigma - z,$$

$$(2.14b) \quad \tau_2 = \zeta_1 + \sigma - z,$$

$$(2.14c) \quad \tau_3 = \frac{t\zeta_4 - y_1 y_2}{z^2} - \frac{t\zeta_3}{z} + t\tau_1 \tau_2 + t y_1 + y_2 + t z(\tau_1 + \tau_2),$$

$$(2.14d) \quad \tau_4 = \frac{t\zeta_4 - (y_1 + \tau_1 z)(t\tau_2 z + y_2)}{z},$$

with

$$(2.15) \quad y_1 = (z - a_1)(z - a_2)y, \quad y_2 = \frac{t(z - a_3)(z - a_4)}{y}.$$

This forms a suitable parameterization of the given linear system of difference equations in terms of three variables, y , z and w . The pair (y, w) are sometimes called the spectral coordinates [7] and w is a gauge factor. It is worth noting that the variable σ plays a role in the asymptotics of the solutions of (1.1a), if we let

$$(2.16) \quad d_1 = \sigma, \quad d_2 = -\sigma - \zeta_1,$$

then d_1 and d_2 are two of the characteristic constants of (1.1a) in [4].

THEOREM 2.3. *The continuous isomonodromic deformation of the linear system specified by (1.1a) with $A(x)$ given by (2.13), (2.14) and (2.15) in t requires that $y(t)$ satisfies (P_{VI}) for the parameters*

$$\alpha = \frac{(a_1 - a_2)^2}{2}, \quad \beta = -\frac{(a_3 - a_4)^2}{2},$$

$$\gamma = \frac{(a_1 + a_2 + \sigma)^2}{2}, \quad \delta = -\frac{(a_3 + a_4 + \sigma)(2 + a_3 + a_4 + \sigma)}{2}.$$

PROOF. While there are very few works on the continuous isomonodromic deformations of linear systems of *difference* equation, we follow the work of [1] by parameterizing the isomonodromic deformations via (2.2) where $B(x)$ is linear. The compatibility between equations of the form (1.1a) and (2.2) may be written

$$(2.17) \quad \frac{dA(x)}{dt} + A(x)\mathcal{B}(x) - \mathcal{B}(x+1)A(x) = 0.$$

We find that (2.17) is overdetermined when $B(x)$ is linear, hence, we may write

$$\mathcal{B}(x) = \frac{x}{t} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{t(t-1)} \begin{pmatrix} 0 & w \\ \tau_3 & 0 \end{pmatrix},$$

where the other relations give us

$$(2.18a) \quad z' = \frac{y_1 - y_2}{t(t-1)},$$

$$(2.18b) \quad y' = \frac{y^2(a_1 + a_2 - 2z) + t(a_3 + a_4 - 2z) + y(\sigma - t(\zeta_1 + \sigma - 2z) + 2z)}{t(t-1)},$$

$$(2.18c) \quad \frac{w'}{w} = -\frac{\sigma + t(\zeta_1 + \sigma - z) + z}{t(t-1)}.$$

Eliminating z from this system gives us the result. \square

The equations (1.1a) and (1.1b) for $A(x)$ and $\mathcal{B}(x)$ given by (2.13) and (2.18a) respectively constitutes a difference-differential Lax pair for (P_{VI}) . This result is essentially a different parameterization of the Lax pair that appeared in [1].

The linear system of the form (1.1a) with $A(x)$ given by (2.13) is also connected to a known Lax pair for the discrete version of the fifth Painlevé equation [7]. We consider the discrete version of the fifth Painlevé equation to be the map

$$\begin{pmatrix} a_1 & a_2 & \sigma; y, z \end{pmatrix} \rightarrow \begin{pmatrix} a_1 + 1 & a_2 + 1 & \sigma - 1; \tilde{y}, \tilde{z} \end{pmatrix}$$

where the values of (\tilde{y}, \tilde{z}) are related to (y, z) via

$$(d-P_V) \quad \tilde{z} + z = a_1 + a_2 + 1 - \frac{a_3 t + a_4 t + \sigma t}{\tilde{y} - t} + \frac{a_1 + a_2 + \sigma + 1}{\tilde{y} - 1},$$

$$y\tilde{y} = \frac{t(z - a_3)(z - a_4)}{(z - a_1)(z - a_2)}.$$

This transformation is a Bäcklund transformation of (P_{VI}) which commutes with the time evolution of (P_{VI}) .

The way in which a Lax pair for $(d-P_V)$ arises is that a_1, \dots, a_4, σ appear as variables that change by integer shifts under the action of discrete isomonodromic

deformations [4]. A discrete isomonodromic deformation is a transformation induced by a gauge transformation of the form

$$(2.19) \quad \tilde{Y}(x) = R(x)Y(x),$$

where $R(x)$ is some rational matrix. We expect $\tilde{Y}(x)$ to satisfy an equation of the form

$$\tilde{Y}(x+1) = \tilde{A}(x)\tilde{Y}(x),$$

where $\tilde{A}(x)$ and $A(x)$ are related by

$$(2.20) \quad \tilde{A}(x)R(x) - R(x+1)A(x) = 0.$$

Our nonautonomous difference equation arises from identifying this as a map from the moduli spaces containing $A(x)$ to the moduli space of matrices containing $\tilde{A}(x)$ in some canonical coordinate system.

THEOREM 2.4. *Given (1.1a) with $A(x)$ given by (2.13), the discrete isomonodromic deformation induced by (2.19) when $R(x)$ takes the form*

$$R(x) = \frac{xI + R_0}{(x - a_1 - 1)(x - a_2 - 1)}.$$

is equivalent to $(d\text{-}P_V)$.

PROOF. We may derive the entries of R_0 in a number of ways, we may compute the residues of (2.20) at the points $x = a_1$, $x = a_2$, $x = a_1 + 1$ and $x = a_2 + 1$, or compare the expansions in x around $x = 0$ and $x = \infty$. We find a suitable form of R_0 to be

$$R_0 = \begin{pmatrix} z - 1 - a_1 - a_2 + \frac{t(a_3 + a_4 + \sigma)}{\tilde{y} - t} & \frac{\tilde{w} - w}{t - 1} \\ \frac{\tau_3 \tilde{w} - w \tilde{\tau}_3}{(t - 1)w\tilde{w}} & \frac{t(a_3 + a_4 - 1 + \sigma - z) + (1 + z)\tilde{y}}{t - \tilde{y}} \end{pmatrix}.$$

The overdetermined compatibility of (2.20) results in $(d\text{-}P_V)$ and

$$\frac{\tilde{w}}{w} = \frac{\tilde{y} - 1}{\tilde{y} - t},$$

which is the equation satisfied by the variable encapsulating the gauge freedom. \square

We consider the equations (1.1a) with $A(x)$ given by (2.13) and (2.19) to be a difference-difference Lax pair. This Lax pair was a result of Arinkin and Borodin [2] (see also [7]).

3. Symmetric difference-differential Lax pair

One of the corollaries of the work on moduli spaces of difference equations was the existence of two moduli spaces of systems of linear difference equations of second order with the same structure as the space of initial conditions for (P_{VI}) . One of these moduli spaces corresponded to a difference-differential Lax pair without symmetry, as given by Adler [1], while the other possesses some symmetry with 4 pairs of symmetric singular points. In this section, we provide a parameterization of the resulting Lax pair that gives (P_{VI}) as a system of continuous isomonodromic deformations.

We recently outlined a discrete version of the Garnier system in which the solutions of the associated linear problem are even functions [26]. A small generalization of this is to consider system of difference equations symmetric around some value. We wish to consider these difference equations as pairs of equations of the form

$$(3.1a) \quad Y(-x - \sigma - 1) = A(x)Y(x),$$

$$(3.1b) \quad Y(-x - \sigma) = Y(x),$$

for some $\sigma \in \mathbb{C}$. The combination of (3.1a) and (3.1b) recovers (1.1a), hence, we consider this to be a system of linear difference equations. This extra constraint is motivated by the additional structure that many elliptic hypergeometric functions and elliptic biorthogonal functions possess.

For the system defined by (3.1), by (3.1b), it is easy to show that $A(x)$ must necessarily satisfy the condition

$$(3.2) \quad A(x)A(-x - \sigma - 1) = I.$$

As discussed in [26], we may always write $A(x)$ in the form

$$(3.3) \quad A(x) = B(-x - \sigma - 1)^{-1}B(x),$$

where $B(x)$ is rational. Without loss of generality, we may multiply by scalar factors so that $B(x)$ is a polynomial. Furthermore, gauge transformations of the form (2.19) have the effect of multiplying $B(x)$ only by a factor on the right, whereas $B(x)$ itself is only defined up to multiplication by a factor on the left. In particular, matrix $B(x)$ from (3.3) is subject to constant gauge transformations on the left and the right.

To make a correspondence the previous section we let

$$(3.4) \quad \det B(x) = t(x - a_1)(x - a_2)(x - a_3)(x - a_4).$$

We may use the gauge freedom on the left and the right to simplify all but one entry, which we choose to be the $(2, 1)$ entry. This means that we may generally take $B(x)$ to be of the form

$$(3.5) \quad B(x) = \begin{pmatrix} (x - \lambda_1)(x - z) + y_1 & w(x - z) \\ \frac{b_0 + b_1x + b_2x^2 + b_3x^3}{w} & (x - \lambda_2)(x - z) + y_2 \end{pmatrix},$$

where the b_i and y_j for $i = 1, 2, 3, 4$ and $j = 1, 2$ are specified up to one parameter by (3.4). These values are

$$\begin{aligned} b_0 &= \frac{t\zeta_4 - (y_1 + z\lambda_1)(y_2 + z\lambda_2)}{z}, \\ b_1 &= t\zeta_1z - t\zeta_2 - tz^2 + \lambda_1\lambda_2 + y_1 + y_2 + (\lambda_1 + \lambda_2)z, \\ b_2 &= t(\zeta_1 - z) - \lambda_1 - \lambda_2 - z, \\ b_3 &= 1 - t, \end{aligned}$$

where we have used the same elementary symmetric functions, ζ_1, \dots, ζ_4 , from the previous section. By suitable gauge transformations, we may take $\lambda_1 = \lambda_2 = 0$, since they are constant with respect to the continuous isomonodromic deformations. We do not simplify in this way because they satisfy some difference equation with respect to the discrete isomonodromic deformations. They essentially play the same

role as w in the previous section, i.e., they are variables encapsulating some left and right gauge freedom.

To obtain (P_{VI}) we let

$$y_1 = (z - a_1)(z - a_2)y, \quad y_2 = \frac{t(z - a_3)(z - a_4)}{y}.$$

These values of b_i and y_j for $i = 0, \dots, 4$ and $j = 1, 2$ are sufficient to ensure that (3.4) is satisfied.

THEOREM 3.1. *The continuous isomonodromic deformations of (1.1a) where $A(x)$ is given by (3.3) and (3.5) is equivalent to $y(t)$ satisfying (P_{VI}) for*

$$\begin{aligned} \alpha &= \frac{(a_1 - a_2)^2}{2}, & \beta &= -\frac{(a_3 - a_4)^2}{2}, \\ \gamma &= \frac{(a_1 + a_2 + \sigma)^2}{2}, & \delta &= -\frac{(a_3 + a_4 + \sigma)(2 + a_3 + a_4 + \sigma)}{2}, \end{aligned}$$

PROOF. The goal is to relate the isomonodromic deformations to an evolution equation for $B(x)$, which, if done correctly, preserves the symmetry condition for $A(x)$. In terms of $Y(x)$, the isomonodromic deformations are specified by an equation of the form

$$(3.6) \quad \frac{dY(x)}{dt} = B_r(x)Y(x),$$

where $B_r(x)$ must satisfy $B_r(x) = B_r(-\sigma - x)$ due to the symmetry condition. We also note that $B(x)$ is only determined up to scalar matrix multiplication on the left, by some factor $B_l(x)$ satisfying $B_l(x) = B_l(-x - \sigma - 1)$. These two conditions result in a consistency condition in which we have matrices acting upon the left and right of $B(x)$. The resulting symmetric analogue of (2.17) is

$$(3.7) \quad \frac{dB(x)}{dt} = B_l(x)B(x) - B(x)B_r(x).$$

When $B_l(x)$ and $B_r(x)$ are linear in $x(x + \sigma)$ and $x(x + \sigma + 1)$ we have an overdetermined system with $B_l(x)$ and $B_r(x)$ given by

$$(3.8) \quad B_l(x) = \begin{pmatrix} 0 & 0 \\ \frac{(1 + \sigma)\lambda_2}{tw} - \frac{x(x + \sigma + 1)}{tw} & \frac{\sigma + 1}{t} - \frac{w'}{w} \end{pmatrix} + \frac{1}{1 - t} \begin{pmatrix} a_1 + a_2 - z - \frac{\lambda_2}{t} & -\frac{w}{t} \\ -\frac{b_1 + (b_2 + b_3z)(z + \lambda_2)}{tw} - \frac{b_3z'}{w} & z - a_3 - a_4 + \frac{\lambda_2}{t} \end{pmatrix},$$

$$(3.9) \quad B_r(x) = \begin{pmatrix} \frac{\sigma}{t} & 0 \\ \frac{\sigma\lambda_1}{tw} - \frac{x(x + \sigma)}{tw} & -\frac{w'}{w} \end{pmatrix} - \frac{1}{1 - t} \begin{pmatrix} a_3 + a_4 - z - \frac{\lambda_1}{t} & \frac{w}{t} \\ \frac{b_1 + (b_2 + b_3z)(z + \lambda_1)}{tw} - \frac{b_3z'}{w} & z - a_1 - a_2 + \frac{\lambda_1}{t} \end{pmatrix},$$

and the remaining conditions in (3.7) give us

$$(3.10) \quad z' = \frac{y_1 - y_2}{t(t-1)},$$

$$(3.11) \quad y' = \frac{1}{t(t-1)} (y^2(a_1 + a_2 - 2z) + t(a_3 + a_4 - 2z) + y(\sigma - t(a_1 + a_2 + a_3 + a_4 + \sigma - 2z) + 2z)).$$

These two equations coincides precisely with (2.18) from the previous section. Following the same steps, we eliminate z to show that y satisfies (P_{VI}) for the given choice of parameters. \square

The above shows that (3.1a), (3.1b) and (3.6) constitutes a Lax pair for (P_{VI}) . However, due to the way in which we factor $A(x)$, the consistency condition, (3.7), is more naturally expressed in terms of $B(x)$ rather than $A(x)$.

One of the most interesting aspects of the symmetric Lax pairs is the existence of classes of symmetries that do not appear naturally in the work of Schlesinger transformations [14]. It is very natural to ask what the analogue of the Schlesinger transformations look like in the symmetric setting, in particular, how does $(d-P_V)$ arise as a discrete isomonodromic deformation of a symmetric system of difference equations.

We find that the evolution of $(d-P_V)$ is expressible in terms of the composition of three transformations, none of which simply shift the variables in the same way as the Schlesinger transformations in the asymmetric setting. It should be noted that while the composition of the transformations commutes with the time evolution of (P_{VI}) . It would be interesting to relate these transformations to a set of known generators of the associated affine Weyl group $D_4^{(1)}$.

One interesting consequence of relating the parameters of $(d-P_V)$ to (3.1a) and (3.1b) is that the value of σ is shifted with evolution of $(d-P_V)$, signifying that we seek a transformation in which the resulting symmetry is changed. Suppose we consider a transformation of the form (2.19), where $R(x) = B(-x - \sigma)$. If $A(x)$ is given by (3.3), then by using (2.20) we find that $\tilde{A}(x)$ is given by

$$\tilde{A}(x) = B(x)B(-x - \sigma)^{-1}.$$

This transformation changes the symmetry of the system as we notice that $\tilde{A}(x)$ satisfies

$$\tilde{A}(x)\tilde{A}(-x - \tilde{\sigma} - 1) = I.$$

If we denote the new value of σ by $\tilde{\sigma}$, then the above indicates that $\tilde{\sigma} = \sigma - 1$. By suitably multiplying by the determinant, we have that the relevant transformation is given by

$$\begin{aligned} \tilde{B}(x) = & t(x + \sigma + a_1)(x + \sigma + a_2) \\ & (x + \sigma + a_3)(x + \sigma + a_4)B(-x - \sigma)^{-1}, \end{aligned}$$

which also corresponds to the adjoint. This means that the effect on the parameters, y and z , are given by

$$(S_1) \quad \begin{pmatrix} a_1 & a_2 & \sigma; y, z \\ a_3 & a_4 & \end{pmatrix} \rightarrow \begin{pmatrix} -a_1 - \sigma & -a_2 - \sigma & \sigma - 1; \frac{t(z-a_3)(z-a_4)}{y(z-a_1)(z-a_2)}, -\sigma - z \end{pmatrix}.$$

The auxiliary variables, λ_1 and λ_2 satisfy $\tilde{\lambda}_1 = -\sigma - \lambda_2$ and $\tilde{\lambda}_2 = -\sigma - \lambda_1$ respectively.

For the second transformation, we consider a matrix acting on the left of $B(x)$. As we explored in a recent paper on discrete Garnier systems, we seek a transformation in which we take two parameters, in the case that is relevant, we take a_3 and a_4 , and send them to $-a_3 - \sigma - 1$ and $-a_4 - \sigma - 1$ respectively. Note that this matrix is easily calculated from the Lax equation

$$(3.12) \quad (x - a_3)(x - a_4)\tilde{B}(x) = R_l(x)B(x),$$

which is induced by a matrix $R_l(x)$ with the property that $R_l(x) = R_l(-x - \sigma - 1)$. This calculation shows

$$R_l(x) = \begin{pmatrix} (x - a_4)(x + \sigma + 1 + a_4) & 0 \\ 0 & (x - a_3)(x + \sigma + 1 + a_3) \end{pmatrix} + (z - \tilde{z}) \begin{pmatrix} z - a_3 - \frac{y(a_4 - \lambda_2)}{t} & \frac{wy}{t} \\ -\frac{t}{wt} \left(z - a_3 - \frac{y(a_4 - \lambda_2)}{t} \right) \left(z - a_4 - \frac{y(a_3 - \lambda_2)}{t} \right) & a_3 - z + \frac{y(a_4 - \lambda_2)}{t} \end{pmatrix}$$

which then induces the transformation

$$(S_2) \quad \begin{pmatrix} a_1 & a_2 & \sigma; y, z \\ a_3 & a_4 & \end{pmatrix} \rightarrow \begin{pmatrix} a_1 & a_2 & \sigma; y, z + \frac{t(1 + \sigma + a_3 + a_4)}{y - t} \\ -a_3 - \sigma - 1 & -a_4 - \sigma - 1 & \end{pmatrix}.$$

where the variables λ_1 and λ_2 become $\lambda_1 + y(z - \tilde{z})$ and $\lambda_2 + t^{-1}y(z - \tilde{z})$ respectively where by \tilde{z} we mean the transformed value of z .

The last transformation to form $(d\text{-P}_V)$ is a transformation that acts on the right of $B(x)$. We take a_1 and a_2 and send these values to $-a_1 - \sigma$ and $-a_2 - \sigma$ respectively. The relevant Lax equation for this transformation is given by

$$(3.13) \quad (x - a_1)(x - a_2)\tilde{B}(x) = B(x)R_r(x),$$

where $R_r(x) = R_r(-x - \sigma)$. It is easy to calculate this matrix from the properties of $B(x)$ and (3.13). This calculation leads us to

$$R_r(x) = \begin{pmatrix} (x - a_1)(x + \sigma + a_1) & 0 \\ 0 & (x - a_2)(x + \sigma + a_2) \end{pmatrix} + (z - \tilde{z}) \begin{pmatrix} a_1 - z + \frac{a_2 - \lambda_1}{y} & \frac{w}{y} \\ -\frac{y}{w} \left(a_1 - z + \frac{a_2 - \lambda_1}{y} \right) \left(a_2 - z + \frac{a_1 - \lambda_1}{y} \right) & z - a_1 - \frac{a_2 - \lambda_1}{y} \end{pmatrix},$$

with the transformation being

$$(S_3) \quad \begin{pmatrix} a_1 & a_2 & \sigma; y, z \\ a_3 & a_4 & \end{pmatrix} \rightarrow \begin{pmatrix} -a_1 - \sigma & -a_2 - \sigma & \sigma; y, z - \frac{(\sigma + a_1 + a_2)y}{y-1} \\ a_3 & a_4 & \end{pmatrix}.$$

This also changes λ_1 and λ_2 to $\lambda_1 - \sigma - a_1 - a_2 + z - \tilde{z}$ and $\lambda_2 - t(\sigma - a_1 - a_2 + z - \tilde{z})$ respectively.

COROLLARY 3.2. *The discrete version of the fifth Painlevé equation arises as a discrete isomonodromic deformation, (2.2), of (1.1a) with $A(x)$ given by (3.3) and (3.5) for*

$$R(x) = (R_r(x)^{S_1})^{-1} B(-x - \sigma),$$

where $R_r(x)^{S_1}$ denotes the value of $R_r(x)$ with the action of S_1 applied to the entries.

PROOF. By using (2.20) and (3.3) we have that

$$(3.14) \quad \tilde{A}(x) = R(x+1)B(-x - \sigma - 1)^{-1}B(x)R(x)^{-1}$$

$$(3.15) \quad = (R_r(x+1)^{S_1})^{-1}B(x)B(-x - \sigma)^{-1}R_r(x)^{S_1},$$

however, we use the fact that

$$B(-x - \sigma)^{-1} = \frac{B(x)^{S_1}}{(x + a_1 + \sigma)(x + a_2 + \sigma)(x + a_3 + \sigma)(x + a_4 + \sigma)},$$

and note that $R_r(x)^{S_1}$ is symmetric around $\sigma - 1$ (as opposed to σ). This means the above may be written as

$$\begin{aligned} \tilde{A}(x) &= \frac{(x - a_1)(x - a_2)(x - a_3)(x - a_4)}{(x + a_1 + \sigma)(x + a_2 + \sigma)(x + a_3 + \sigma)(x + a_4 + \sigma)} \\ &\quad (B(-x - \sigma)^{S_1} R_r(-x - \sigma)^{S_1})^{-1} (B(x)^{S_1} R_r(x)^{S_1}), \end{aligned}$$

where $B(x)^{S_1} R_r(x)^{S_1} = B(x)^{S_3 \circ S_1}$, giving

$$\frac{(x - a_3)(x - a_4)}{(x + a_3 + \sigma)(x + a_4 + \sigma)} (B(-x - \sigma)^{S_3 \circ S_1}) B(x)^{S_3 \circ S_1}.$$

We now note that this formulation is only defined up to some left multiplication by $R_l(x)^{S_1 \circ S_3}$ (which now satisfies $R_l(x)^{S_1 \circ S_3} = R_l(-\sigma - x)^{S_1 \circ S_3}$), which recovers the remaining part of the calculation, showing

$$\tilde{A}(x) = (B(-x - \sigma)^{S_2 \circ S_3 \circ S_1})^{-1} B(x)^{S_2 \circ S_3 \circ S_1}.$$

Since each transformation, (S_1) , (S_2) and (S_3) , has a simple effect on the parameters y and z , it is easy to see that $(d\text{-}P_V)$ arises as the composition of (S_1) followed by (S_3) and then (S_2) . \square

4. The list of symmetric Lax pairs

We simplify the above situation a little by letting $\sigma = 0$ and by letting $\lambda_1 = \lambda_2 = 0$. These simplifications are perfectly valid for considering the difference differential Lax pairs. These values only change when one is calculating the symmetries, as we have done for (P_{VI}) above but we do not present for the cases below.

Under these simplifying assumptions, each of the Lax pairs we present takes the form (1.1a) in which $A(x)$ is given by

$$A(x) = B(-x - 1)^{-1} B(x),$$

where $B(x)$ takes the form

$$(4.1) \quad B(x) = \begin{pmatrix} x(x-z) + y_1 & x-z \\ b_0 + b_1x + b_2x^2 + b_3x^3 & x(x-z) + y_2 \end{pmatrix},$$

where the b_i and y_j for $i = 1, 2, 3, 4$ and $j = 1, 2$ are specified up to one parameter by a determinant.

4.1. A Lax pair for (P_V) . The corresponding Lax pair for (P_V) is specified by

$$\det B(x) = t(x - a_1)(x - a_2)(x - a_3).$$

This gives linear conditions on each of the b_i , which may be solved to give

$$\begin{aligned} b_0 &= -t((a_2 - z)(a_3 - z) + a_1(a_2 + a_3 - z)), \\ b_1 &= t(a_2 + a_3) + y_1 + yy_2, \\ b_2 &= -t - z, & b_3 &= 1. \end{aligned}$$

Our choice for y_1 and y_2 are given by

$$(4.2) \quad y_1 = (1 - y)(z - a_2)(z - a_3), \quad y_2 = \frac{t(z - a_1)}{1 - y}.$$

The isomonodromic deformations are also given by (3.7) where

$$(4.3) \quad B_l(x) = \begin{pmatrix} 0 & -\frac{1}{t} \\ -\frac{x(x+1) + t(a_2 + a_3 - z) + (y+1)y_2}{t} & 1 + \frac{1}{t} \end{pmatrix},$$

$$(4.4) \quad B_r(x) = \begin{pmatrix} 1 & -\frac{1}{t} \\ -\frac{x^2 + t(a_2 + a_3 - z) + 2y_1 + (y-1)y_2}{t} & 0 \end{pmatrix}.$$

From (3.7) we obtain

$$(4.5a) \quad y' = \frac{(a_2 + a_3)(y - 1)^2 - ty}{t} - \frac{2(y - 1)yz}{t},$$

$$(4.5b) \quad z' = \frac{y_2 - y_1}{t}.$$

The system (4.5) written in terms of y alone give (P_V) for the parameters

$$(4.6) \quad \alpha = \frac{(a_2 - a_3)^2}{2}, \quad \beta = -\frac{(a_2 + a_3)^2}{2}, \quad \gamma = -1 - 2a_1.$$

4.2. A Lax pair for $(P_{III}(D_6))$. To make a correspondence with $(P_{III}(D_6))$, we find that the relevant time variable is given by the square root of the leading term, i.e., our associated linear problem is still specified by a matrix, $B(x)$, of the form (4.1) where we take the determinant to be

$$\det B(x) = t^2(x - a_1)(x - a_2).$$

which specifies linear conditions for b_0, \dots, b_3 which are satisfied by

$$\begin{aligned} b_0 &= t^2(a_1 + a_2 - z), & b_1 &= y_1 + y_2 - t^2, \\ b_2 &= -z, & b_3 &= 1, \end{aligned}$$

where y_1 and y_2 are

$$(4.7) \quad y_1 = ty(z - a_2), \quad y_2 = \frac{t(z - a_1)}{y}.$$

The relevant compatibility condition is given by (3.7) where

$$(4.8) \quad B_l(x) = \begin{pmatrix} 0 & -\frac{2}{t} \\ \frac{2(t^2 - x(x+1) - 2y_2)}{t} & \frac{2}{t} \end{pmatrix},$$

$$(4.9) \quad B_r(x) = \begin{pmatrix} 0 & -\frac{2}{t} \\ -\frac{2x^2 - 2t^2 + 4y_1}{t} & 0 \end{pmatrix}.$$

Computing the entries of (3.7) reveals the system of first order equations

$$(4.10a) \quad y' = \frac{y(2ty - 4z - 1) + 2t}{t},$$

$$(4.10b) \quad z' = \frac{2(y_2 - y_1)}{t},$$

which is equivalent to y satisfying $(P_{III}(D_6))$ for

$$(4.11) \quad \alpha = -8a_2, \quad \beta = 4(1 + 2a_1), \quad \gamma = 4, \quad \delta = -4.$$

4.3. A Lax pair for $(P_{III}(D_7))$. The associated linear problem is of the form (4.1) where

$$\det B(x) = t(x - a_1).$$

This condition determines that b_0, \dots, b_3 are

$$b_0 = -t, \quad b_1 = \frac{ta_1 - y^2 - tz}{y}, \quad b_2 = -z, \quad b_3 = 1,$$

with

$$(4.12) \quad y_1 = -y, \quad y_2 = -\frac{t(z - a_1)}{y}.$$

The left and right transformation matrices in (3.7) are

$$(4.13) \quad B_l(x) = \begin{pmatrix} 0 & -\frac{1}{t} \\ \frac{2tz - x(x+1)y - 2ta_1}{ty} & \frac{1}{t} \end{pmatrix},$$

$$(4.14) \quad B_r(x) = \begin{pmatrix} 0 & -\frac{1}{t} \\ \frac{2y - x^2}{t} & 0 \end{pmatrix}.$$

Using these matrices in (3.7) results in the following system of first order differential equations

$$(4.15a) \quad y' = -1 - \frac{2yz}{t},$$

$$(4.15b) \quad z' = \frac{y_2 - y_1}{t},$$

which implies that y satisfies $(P_{III}(D_7))$ for the parameter

$$\beta = -4 - 8a_1.$$

4.4. A Lax pair for $(P_{III}(D_8))$. The associated linear problem for this last case is of the form (4.1) where we make a correspondence with $(P_{III}(D_8))$ when

$$\det B(x) = \frac{t}{4}.$$

Comparing coefficients imposes enough constraints to determine that b_0, \dots, b_3 are

$$b_0 = 0, \quad b_1 = \frac{t + y^2}{2y}, \quad b_2 = -z, \quad b_3 = 1$$

with y being defined by

$$y_1 = \frac{y}{2}, \quad y_2 = \frac{t}{2y}.$$

Under this choice we determine that $R_l(x)$ and $R_r(x)$ are

$$(4.16) \quad B_l(x) = \begin{pmatrix} 0 & -\frac{1}{t} \\ -\frac{x(x+1)}{t} - \frac{1}{y} & \frac{1}{t} \end{pmatrix},$$

$$(4.17) \quad B_r(x) = \begin{pmatrix} 0 & -\frac{1}{t} \\ -\frac{x^2}{t} - \frac{y}{t} & 0 \end{pmatrix}.$$

These matrices, when used in (3.7), give the system

$$(4.18a) \quad y' = -\frac{2yz}{t},$$

$$(4.18b) \quad z' = \frac{y_2 - y_1}{t}.$$

5. A symmetric version of Painlevé VI

The aim of this section is to determine a version of (P_{VI}) . For this, we simply take a slightly different form of $B(x)$ from the previous sections, in which

$$(5.1) \quad B(x) = \begin{pmatrix} x^2 - z^2 + y & x - z \\ b_0 + b_1x + b_2x^2 + b_3x^3 & \frac{x^2 - z^2}{t} + y_2 \end{pmatrix}.$$

The coefficients of that determinant by

$$(5.2) \quad \det B(x) = t \sum_{k=0}^4 m_k x^k$$

so that the coefficients are symmetric in the roots of $\det B(x)$. Following the same steps as above we have

$$\begin{aligned} b_0 &= \frac{m_0 t + (y - z^2)(z^2 - y_2)}{z}, \\ b_1 &= \frac{m_1 t z + m_0 t + (y - z^2)(z^2 - y_2)}{z^2}, \\ b_2 &= z - t(m_4 z + m_3), \\ b_3 &= 1 - t m_4, \end{aligned}$$

where

$$y_2 = \frac{\det B(z)}{y}.$$

With this choice, we find that the left and right matrices are

$$(5.3) \quad B_l(x) = \begin{pmatrix} \frac{z}{t} & -\frac{1}{tb_3} \\ \frac{tm_2 - z(b_2 + b_3 - 2z) - y - y_2}{tb_3} - \frac{x(x+1)}{t} & \frac{b_3 - b_2}{b_3 t} \end{pmatrix},$$

$$(5.4) \quad B_r(x) = \begin{pmatrix} -\frac{b_2}{tb_3} & -\frac{1}{tb_3} \\ \frac{tm_2 - z(b_2 - 2z) - y - y_2}{tb_3} - \frac{x^2}{t} & \frac{z}{t} \end{pmatrix},$$

whose compatibility results in the following version of (P_{VI})

$$(5.5a) \quad y' = \frac{1}{m_4 t - 1} \left(y \left(m_3 + 2 \left(\frac{1}{t} + m_4 \right) z \right) - \frac{1}{t} \frac{d}{dx} \det B(x) \Big|_{x=z} \right),$$

$$(5.5b) \quad z' = \frac{1}{m_4 t - 1} \left(\frac{y}{t} - \frac{\det B(z)}{ty} \right).$$

First of all, we recover (P_{VI}) if we make the substitution $Y = (z - a_1)(z - a_2)/y$ where a_1 and a_2 are roots of $\det B(x)$, then we obtain a symmetric version of (P_V) when $m_4 = 0$, and a symmetric version of $(P_{III}(D_6))$ when $m_4 = m_3 = 0$ and so on. At each stage there is a different transformation that relates the resulting equations to the versions of (P_V) to $(P_{III}(D_8))$ that appear in the introduction.

6. Discussion

One of the interesting features of this work is that the symmetric Lax pairs seem to have a larger group of symmetries than that of the non-symmetric cases. For the non-symmetric cases, the relevant group acting on the parameter space has the structure of a lattice, whereas in the symmetric case, we have a group generated by a pairs of transformations, each pair is the generator of some infinite dihedral group. This setting is much closer to the affine Weyl symmetries of the Painlevé equations [21] and the discrete Painlevé equations [22].

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